A Refinement of a Theorem of Diaconis-Evans-Graham

Lora R. Du[1] and Kathy Q. Ji[2]

[1,2] Center for Applied Mathematics and KL-AAGDM

Tianjin University

Tianjin 300072, P.R. China

Emails: [1]loradu@tju.edu.cn and [2]kathyji@tju.edu.cn

Abstract: The note is dedicated to refining a theorem by Diaconis, Evans, and Graham concerning successions and fixed points of permutations. This refinement specifically addresses non-adjacent successions, predecessors, excedances, and drops of permutations.

Keywords: permutations, successions, fixed points, excedances, bijection

AMS Classification: 05A15, 05A19

1 Introduction

The main objective of this paper is to give a refinement of a theorem of Diaconis-Evans-Graham [4] on successions and fixed points of permutations.

Let $\mathfrak{S}_n$ be the set of permutations on $[n] = \{1, 2, \ldots, n\}$. For a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$, an index $1 \leq i \leq n-1$ is called a succession if $\sigma_i + 1 = \sigma_{i+1}$, whereas an index $1 \leq i \leq n$ is called a fixed point if $\sigma_i = i$. Let $\mathrm{Suc}(\sigma)$ be the set of successions of σ, that is

$$\mathrm{Suc}(\sigma) = \{1 \leq i \leq n-1 \mid \sigma_i + 1 = \sigma_{i+1}\}$$

and let $\overline{\mathrm{Fix}}(\sigma)$ denote the set of fixed points of σ distinct from n. To wit,

$$\overline{\mathrm{Fix}}(\sigma) = \{1 \leq i \leq n-1 \,|\sigma_i = i\}.$$

It should be noted that the index n is excluded in the definition of $\overline{\mathrm{Fix}}(\sigma)$.

Given a subset $I \subseteq [n-1]$, let $\mathrm{Suc}_n(I)$ be the set of permutations σ of $[n]$ such that $\mathrm{Suc}(\sigma) = I$ and let $\overline{\mathrm{Fix}}_n(I)$ be the set of permutations $\sigma \in \mathfrak{S}_n$ such that $\overline{\mathrm{Fix}}(\sigma) = I$.

Diaconis, Evans and Graham [4] discovered the following beautiful result.

Theorem 1.1. (Diaconis-Evans-Graham) *Let $n \geq 1$ and $I \subseteq [n-1]$. Then there is a bijection between $\mathrm{Suc}_n(I)$ and $\overline{\mathrm{Fix}}_n(I)$.*

It is worth mentioning that Chen [2] provided a bijective proof of the Diaconis-Evans-Graham theorem for the case $I = \emptyset$ via the first fundamental transformation. Brenti and Marietti [1] extended this result within the context of colored permutations in the complex reflection groups $G(r, p, n)$ where r, p, n are positive integers with p dividing n. Recently, Chen and Fu [3] established a left succession analogue of the Diaconis-Evans-Graham theorem, exemplifying the idea of a grammar assisted bijection. Additionally, Ma, Qi, Yeh and Yeh [5] utilized the grammatical labeling technique to demonstrate that two triple set-valued statistics of permutations are quidistributed on symmetric groups. This implies that the number of permutations in $\mathfrak{S}_n$ with the given set I of fixed points distinct from 1 equals to the number of permutations in $\mathfrak{S}_n$ having I as a set of σ_{i+1} such that $\sigma_i + 1 = \sigma_{i+1}$ for $1 \leq i \leq n-1$.

Inspired by a recent work of Chen and Fu [3], we discover a refinement of the Diaconis-Evans-Graham theorem involving two variations of successions, that is, non-adjacent successions and predecessors. Recall that Diaconis, Evans, and Graham refer to a succession of $\sigma = \sigma_1 \cdots \sigma_n$ as an unseparated pair $(k, k+1)$ of σ provided that $\sigma_k + 1 = \sigma_{k+1}$. This terminology and the motivation for studying this concept stem from regarding a permutation as the outcome of shuffling a deck of n cards. The succession has also been extensively studied in the literature, see, e.g., [1, 3, 5–9] , and the references cited there.

Definition 1.2 (Non-adjacent succession)**.** *Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$, an index i $(1 \leq i \leq n-2)$ is called a non-adjacent succession of σ if there exists an integer $i+2 \leq j \leq n$ such that $\sigma_j = \sigma_i + 1$. The set of non-adjacent successions of σ is denoted by* $\mathrm{najSuc}(\sigma)$.

Definition 1.3 (Predecessor)**.** *Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$, an index i $(2 \leq i \leq n)$ is called a predecessor of σ if there exists an integer $1 \leq j < i$ such that $\sigma_j = \sigma_i + 1$. The set of predecessors of σ is denoted by* $\mathrm{Pred}(\sigma)$.

For the permutation $\sigma = 4\,1\,2\,6\,7\,5\,3$, we see that

$$\mathrm{Suc}(\sigma) = \{2, 4\}, \quad \mathrm{najSuc}(\sigma) = \{1, 3\}, \quad \text{and} \quad \mathrm{Pred}(\sigma) = \{6, 7\}.$$

To state our refinement, we also need to recall an excedance and a drop of a permutation. For a permutation $\sigma \in \mathfrak{S}_n$, an index $1 \leq i \leq n$ is called an excedance if $\sigma_i > i$ and an index $1 \leq i \leq n$ is called a drop if $\sigma_i < i$. Define

$$\overline{\mathrm{Drop}}(\sigma) = \{\sigma_i \mid 1 \leq i \leq n-1, \sigma_i < i\},$$

$$\overline{\mathrm{Exc}}(\sigma) = \{\sigma_i \mid 1 \leq i \leq n-1, \sigma_i > i\}.$$

It should be noted that the index n is excluded in the definition of $\overline{\mathrm{Drop}}(\sigma)$ and the set $\overline{\mathrm{Exc}}(\sigma)$. We have the following result.

Theorem 1.4. *For $n \geq 1$, there is a bijection ϕ between $\mathfrak{S}_n$ and $\mathfrak{S}_n$ such that for $\sigma \in \mathfrak{S}_n$ and $\tau = \phi(\sigma)$, we have*

$$\overline{\mathrm{Fix}}(\sigma) = \mathrm{Suc}(\tau), \quad \overline{\mathrm{Drop}}(\sigma) = \mathrm{najSuc}(\tau) \quad \textit{and} \quad \overline{\mathrm{Exc}}(\sigma) = \mathrm{Pred}(\tau). \tag{1.1}$$

Proof. Given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$, we define $\tau = \phi(\sigma)$ via three steps:

Step 1. Define $\overline{\sigma} = \overline{\sigma}_1\overline{\sigma}_2\cdots\overline{\sigma}_n$, where for $1 \leq i \leq n$,

$$\overline{\sigma}_i = n + 1 - \sigma_{n-i+1}.$$

Step 2. Let $\hat{\sigma} = \hat{\sigma}_1\hat{\sigma}_2\cdots\hat{\sigma}_n$, where $\hat{\sigma}_i = \overline{\sigma}_{i+1}$, for $1 \leq i \leq n-1$ and $\hat{\sigma}_n = \overline{\sigma}_1$. Then we write $\hat{\sigma}$ in cycle form $(a_1, a_2, \ldots, a_r)(b_1, b_2, \ldots, b_s)\cdots(c_1, c_2, \ldots, c_t)$, where

- the first cycle is the cycle including n, where n is placed as the last element in this cycle;
- other cycles are written with its smallest element first and the cycles are written in decreasing order of their smallest element.

Define $\overline{\tau}$ to be the permutation obtained from $\hat{\sigma}$ by writing it in the above cycle form and erasing the parentheses. It can be easily verified that $\hat{\sigma}$ can be uniquely reconstructed from $\overline{\tau}$. To achieve this, we begin by inserting the first left parenthesis before $\overline{\tau}_1$ and the first right parenthesis after n. Then, we insert a left parenthesis before each left-to-right minimum occurring after n in $\overline{\tau}$. Finally, we place a right parenthesis preceding each internal left parenthesis and at the end to obtain $\hat{\sigma}$.

Step 3. Take the inversion of $\overline{\tau}$, denoted by $\overline{\tau}^{-1} = \overline{\tau}_1^{-1}\cdots\overline{\tau}_n^{-1}$. Define

$$\tau = \phi(\sigma) = \tau_1\cdots\tau_n, \quad \text{where } \tau_i = n + 1 - \overline{\tau}^{-1}_{n-i+1} \quad \text{for } 1 \leq i \leq n.$$

We proceed to demonstrate that σ and $\tau = \phi(\sigma)$ satisfy the relations (1.1).

Let

$$k \in \overline{\mathrm{Fix}}(\sigma), \quad \sigma_r \in \overline{\mathrm{Drop}}(\sigma), \quad \text{and} \quad \sigma_s \in \overline{\mathrm{Exc}}(\sigma).$$

By definition, we see that $\sigma_k = k$, $\sigma_r < r$ and $\sigma_s > s$. Moreover, $k, r, s \neq n$.

Set $K = n + 1 - k$, $R = n + 1 - r$ and $S = n + 1 - s$. Since $k, r, s \neq n$, we see that $K, R, S \neq 1$.

From the construction of the first step of the bijection ϕ, we see that

$$\overline{\sigma}_K = K, \quad \overline{\sigma}_R = n+1-\sigma_r > R, \quad \text{and} \quad \overline{\sigma}_S = n+1-\sigma_s < S.$$

Moreover, according to the construction of the second step of the bijection ϕ, we have

$$\hat{\sigma}_{K-1} = \overline{\sigma}_K = K, \quad \hat{\sigma}_{R-1} = \overline{\sigma}_R > R, \quad \text{and} \quad \hat{\sigma}_{S-1} = \overline{\sigma}_S < S. \tag{1.2}$$

If we write the cycle decomposition of $\hat{\sigma}$ in the cycle representation described above, then there will be a cycle of the form $(\ldots, K-1, K, \ldots)$. After the parentheses are removed to form $\overline{\tau}$, we will have $\overline{\tau}_j = K-1$ and $\overline{\tau}_{j+1} = K$ for some $1 \le j \le n-1$. Hence $\overline{\tau}^{-1}_{K-1} = j$, $\overline{\tau}^{-1}_K = j+1$, and so

$$\tau_k = n+1-\overline{\tau}^{-1}_{n+1-k} = n-j \quad \text{and} \quad \tau_{k+1} = n+1-\overline{\tau}^{-1}_{n-k} = n+1-j.$$

It follows that $k \in \mathrm{Suc}(\tau)$.

We proceed to show that $\sigma_r \in \mathrm{najSuc}(\tau)$. Similarly, under the assumption of the cycle form, there will be a cycle of the form $(\ldots, R-1, \overline{\sigma}_R, \ldots)$ in the cycle representation of $\hat{\sigma}$. After the parentheses are removed to form $\overline{\tau}$, we will have $\overline{\tau}_i = R-1$ and $\overline{\tau}_{i+1} = \overline{\sigma}_R > R$ for some $1 \le i \le n-1$. Hence $\overline{\tau}^{-1}_{R-1} = i$, $\overline{\tau}^{-1}_{\overline{\sigma}_R} = i+1$, and so

$$\tau_{r+1} = n+1-\overline{\tau}^{-1}_{R-1} = n+1-i \quad \text{and} \quad \tau_{n+1-\overline{\sigma}_R} = n+1-\overline{\tau}^{-1}_{\overline{\sigma}_R} = n-i.$$

Since $n+1-\overline{\sigma}_R = \sigma_r < r$, we derive that $\sigma_r \in \mathrm{najSuc}(\tau)$.

It remains to show that $\sigma_s \in \mathrm{Pred}(\tau)$. By (1.2), we see that $\hat{\sigma}_{S-1} \le S-1$. If we express the cycle decomposition of $\hat{\sigma}$ using the cycle representation described above, then there will be two situations: a cycle of the form $(\ldots, S-1, \hat{\sigma}_{S-1}, \ldots)$ or a cycle of the form $(\hat{\sigma}_{S-1}, \ldots, S-1)$ occurs in the cycle decomposition of $\hat{\sigma}$. In particular, if $\hat{\sigma}_{S-1} = S-1$, then there will be a 1-cycle $(\hat{\sigma}_{S-1})$. This case can be regarded as a special case of the situation where $(\hat{\sigma}_{S-1}, \ldots, S-1)$ occurs.

(a) If a cycle of the form $(\ldots, S-1, \hat{\sigma}_{S-1}, \ldots)$ occurs in the cycle decomposition of $\hat{\sigma}$, then $\overline{\sigma}_S = \hat{\sigma}_{S-1} \le S-2$, and so $\sigma_s \ge s+2$. After the parentheses are removed to obtain $\overline{\tau}$, we will have $\overline{\tau}_t = S-1$ and $\overline{\tau}_{t+1} = \hat{\sigma}_{S-1} = \overline{\sigma}_S \le S-2$ for some $1 \le t \le n-1$. Hence $\overline{\tau}^{-1}_{S-1} = t$, $\overline{\tau}^{-1}_{\overline{\sigma}_S} = t+1$, and so

$$\tau_{s+1} = n+1-\overline{\tau}^{-1}_{S-1} = n+1-t \quad \text{and} \quad \tau_{n+1-\overline{\sigma}_S} = n+1-\overline{\tau}^{-1}_{\overline{\sigma}_S} = n-t.$$

Since $n+1-\overline{\sigma}_S = \sigma_s > s+2$, we derive that $\sigma_s \in \mathrm{Pred}(\tau)$.

(b) If a cycle of the form $(\hat{\sigma}_{S-1}, \ldots, S-1)$ occurs in the cycle decomposition of $\hat{\sigma}$, then the element n is not in this cycle according to the cycle form described above, and so $(\hat{\sigma}_{S-1}, \ldots, S-1)$ lies after the first cycle including n. Erase the parentheses to get $\overline{\tau}$. We will have $\overline{\tau}_{t+1} = \hat{\sigma}_{S-1} = \overline{\sigma}_S$ for some $1 \le t \le n-1$. Since the cycles except for the first cycle are written with its smallest element first and the cycles are written in decreasing order of their smallest element, we deduce that $\overline{\tau}_t > \overline{\tau}_{t+1} = \overline{\sigma}_S$. Assume that $\overline{\tau}_t = T$. Hence $\overline{\tau}^{-1}_T = t$, $\overline{\tau}^{-1}_{\overline{\sigma}_S} = t+1$, and so

$$\tau_{n+1-T} = n+1-\overline{\tau}^{-1}_T = n+1-t \quad \text{and} \quad \tau_{n+1-\overline{\sigma}_S} = n+1-\overline{\tau}^{-1}_{\overline{\sigma}_S} = n-t.$$

Since $\sigma_s = n+1-\overline{\sigma}_S > n+1-T$, we derive that $\sigma_s \in \mathrm{Pred}(\tau)$.

It is straightforward to verify that this process is reversible, and the reversed process also satisfies the relations (1.1). Thus, we complete the proof of the theorem. ∎

Remark. Below is an example of the construction of $\phi(\sigma)$ from the same permutation $\sigma = 7\,2\,6\,4\,1\,3\,5$ given by Diaconis, Evans and Graham in [4, Remark 4.2].

Step 1. We first set $\overline{\sigma} = 3\,5\,7\,4\,2\,6\,1$.

Step 2. We then define $\hat{\sigma} = 5\,7\,4\,2\,6\,1\,3$ and we adopt the following cycle form of $\hat{\sigma}$: $(3\,4\,2\,7)(1\,5\,6)$. Thus, $\overline{\tau} = 3\,4\,2\,7\,1\,5\,6$.

Step 3. Take the inversion of $\overline{\tau}$, denoted by $\overline{\tau}^{-1} = 5\,3\,1\,2\,6\,7\,4$. Let

$$\tau = \phi(\sigma) = 4\,1\,2\,6\,7\,5\,3,$$

which differs from $\widehat{\rho}(\sigma) = 7\,1\,2\,5\,6\,4\,3$ as obtained by Diaconis, Evans and Graham [4] through their bijection.

It is apparent that

$$\overline{\mathrm{Fix}}(\sigma) = \mathrm{Suc}(\tau) = \{2,4\}, \; \overline{\mathrm{Drop}}(\sigma) = \mathrm{najSuc}(\tau) = \{1,3\}, \quad \overline{\mathrm{Exc}}(\sigma) = \mathrm{Pred}(\tau) = \{6,7\}.$$

Declaration of competing interest.

The authors declare that they have no financial, personal, or professional relationships that could be perceived as a conflict of interest.

Data availability.

No data was used for the research described in the article.

Acknowledgment. This work was supported by the National Natural Science Foundation of China.

References

[1] F. Brenti and M. Marietti, Fixed points and adjacent ascents for classical complex reflection groups, Adv. in Appl. Math. 101 (2018) 168–183.

[2] W.Y.C. Chen, The skew, relative, and classical derangements, Discrete Math. 160 (1996) 235–239.

[3] W.Y.C. Chen and A.M. Fu, A grammar of Dumont and a theorem of Diaconis-Evans-Graham, arXiv: 2402.02743.

[4] P. Diaconis, S.N. Evans and R. Graham, Unseparated pairs and fixed points in random permutations, Adv. in Appl. Math., 61 (2014) 102–124.

[5] S.-M. Ma, H. Qi, J. Yeh and Y.-N. Yeh, On the joint distributions of succession and Eulerian statistics, arXiv: 2401.01760v2.

[6] T. Mansour and M. Shattuck, Counting permutations by the number of succcessions within cycles, Discrete Math. 339 (2016) 1368–1376.

[7] J. Reilly and S. Tanny, Counting permutations by successions and other figures, Discrete Math. 32 (1980) 69–76.

[8] D.P. Roselle, Permutations by number of rises and successions, Proc. Amer. Math. Soc. 19 (1968) 8–16.

[9] S. Tanny, Permutations and successions, J. Combin. Theory Ser. A 21 (1976) 196–202.